Assignment 0:
Rule Induction

CSCI 5535 / ECEN 5533: Fundamentals of Programming Languages

Spring 2018: due Friday, January 26, 2018

Go to the course web page to understand the whiteboard policy for collaboration regarding the homework assignments, the late policy regarding timeliness of homework submissions, and the use of Moodle.

Homework will typically consist of a theoretical section and an implementation section. For the first assignment, there is only the theoretical section. You are required to typeset your answers.

In this first assignment we are asking you to practice proving theorems by rule induction. You may find this assignment difficult. Start early, and ask us for help if you get stuck! In particular, you are encouraged to ask clarification questions in class, on the discussion forum, or in office hours (but do *not* post your solutions directly).

Remember to submit early and that **late assignments will not be scored.**

1 Course mechanics

The purpose of this question is to ensure that you get familiar with this course's collaboration policy.

As in any class, you are responsible for following our collaboration policy; violations will be handled according to university policy.

Task 1.1 (4 pts). Our course's collaboration policy is on the course web site. Read it; then, for each of the following situations, decide whether or not the students' actions are permitted by the policy. Explain your answers.

1. Dolores and Toby are discussing Problem 3 by IM. Meanwhile, Toby is writing up his solution to that problem.

2. Amy, Jeff, and Chris split a pizza while talking about their homework, and by the end of lunch, their pizza box is covered with notes and solutions. Chris throws out the pizza box and the three go to class.

3. Ian and Jeremy write out a solution to Problem 4 on a whiteboard in CSEL. Then, they erase the whiteboard and run to the atrium. Sitting at separate tables, each student types up the solution on his laptop.

4. Nitin and Margaret are working on this homework over lunch; they write out a solution to Problem 2 on a napkin. After lunch, Nitin pockets the napkin, heads home, and writes up his solution.

2 Shuffling cards

For this assignment, we will play with cards. Rather than the standard 52 different cards, we will define four different cards, one for each suit. We model a deck of cards as a list.

$$\frac{}{\heartsuit\ \mathsf{card}}\ (1) \qquad \frac{}{\spadesuit\ \mathsf{card}}\ (2) \qquad \frac{}{\clubsuit\ \mathsf{card}}\ (3) \qquad \frac{}{\diamondsuit\ \mathsf{card}}\ (4)$$

$$\frac{}{\mathsf{nil}\ \mathsf{deck}}\ (5) \qquad \frac{c\ \mathsf{card} \quad s\ \mathsf{deck}}{\mathsf{cons}(c, s)\ \mathsf{deck}}\ (6)$$

These rules are an iterated inductive definition for a deck of cards; these rules lead to the following induction principle:

In order to show $\mathcal{P}(s)$ whenever $s\ \mathsf{deck}$, it is enough to show

1. $\mathcal{P}(\mathsf{nil})$
2. $\mathcal{P}(\mathsf{cons}(c, s))$ assuming $c\ \mathsf{card}$ and $\mathcal{P}(s)$

We also want to define an judgment unshuffle. Shuffling takes two decks of cards and creates a new deck of cards by interleaving the two decks in some way; un-shuffling is just the opposite operation.

The definition of $\mathsf{unshuffle}(s_1, s_2, s_3)$ defines a relation between three decks of cards s_1, s_2, and s_3, where s_2 and s_3 are arbitrary "unshufflings" of the first deck – sub-decks where the order from the original deck is preserved, so that the two sub-decks s_2 and s_3 could potentially be shuffled back to produce the original deck s_1.

$$\frac{}{\mathsf{unshuffle}(\mathsf{nil}, \mathsf{nil}, \mathsf{nil})}\ (7) \qquad \frac{c\ \mathsf{card} \quad \mathsf{unshuffle}(s_1, s_2, s_3)}{\mathsf{unshuffle}(\mathsf{cons}(c, s_1), s_2, \mathsf{cons}(c, s_3))}\ (8)$$

$$\frac{c\ \mathsf{card} \quad \mathsf{unshuffle}(s_1, s_2, s_3)}{\mathsf{unshuffle}(\mathsf{cons}(c, s_1), \mathsf{cons}(c, s_2), s_3)}\ (9)$$

Task 2.1 (5 pts). Prove the following (by giving a derivation). There are at least two ways to do so.

$$\mathsf{unshuffle}(\mathsf{cons}(\heartsuit, \mathsf{cons}(\spadesuit, \mathsf{cons}(\spadesuit, \mathsf{cons}(\diamondsuit, \mathsf{nil})))),\quad \mathsf{cons}(\spadesuit, \mathsf{cons}(\diamondsuit, \mathsf{nil})),\quad \mathsf{cons}(\heartsuit, \mathsf{cons}(\spadesuit, \mathsf{nil})))$$

Task 2.2 (5 pts). What was the other way? (describe briefly, or just give the other derivation)

Task 2.3 (10 pts). Prove that unshuffle has the following property:

For all s_1, if $s_1\ \mathsf{deck}$, then there exists s_2 and s_3 such that $\mathsf{unshuffle}(s_1, s_2, s_3)$.

Note that there are a number of different ways of proving this! What the s_2 and s_3 "look like" may be very different depending on how you write the proof. Restate any induction principle you use, and identify what property P you are proving with that induction principle.

Task 2.4 (10 pts). Give an inductive definition of separate, a judgment similar to unshuffle that relates a deck of cards to two "un-shuffled" sub decks where all of the red cards (suits $\diamondsuit$ and $\heartsuit$) are in one deck and all the black cards (suits $\clubsuit$ and $\spadesuit$) are in the other. The following should be provable from your inductive definition:

$$\begin{array}{r}
\mathsf{separate}(\mathsf{cons}(\heartsuit, \mathsf{cons}(\diamondsuit, \mathsf{cons}(\spadesuit, \mathsf{nil}))),\ \ \mathsf{cons}(\heartsuit, \mathsf{cons}(\diamondsuit, \mathsf{nil})),\ \ \mathsf{cons}(\spadesuit, \mathsf{nil})) \\
\mathsf{separate}(\mathsf{cons}(\spadesuit, \mathsf{cons}(\diamondsuit, \mathsf{cons}(\clubsuit, \mathsf{cons}(\heartsuit, \mathsf{nil})))),\ \ \mathsf{cons}(\diamondsuit, \mathsf{cons}(\heartsuit, \mathsf{nil})),\ \ \mathsf{cons}(\spadesuit, \mathsf{cons}(\clubsuit, \mathsf{nil}))) \\
\mathsf{separate}(\mathsf{cons}(\clubsuit, \mathsf{cons}(\heartsuit, \mathsf{cons}(\clubsuit, \mathsf{cons}(\spadesuit, \mathsf{nil})))),\ \ \mathsf{cons}(\heartsuit, \mathsf{nil}),\ \ \mathsf{cons}(\clubsuit, \mathsf{cons}(\clubsuit, \mathsf{cons}(\spadesuit, \mathsf{nil}))))
\end{array}$$

However $\mathsf{separate}(\mathsf{cons}(\heartsuit, \mathsf{cons}(\spadesuit, \mathsf{nil})), \mathsf{cons}(\heartsuit, \mathsf{cons}(\spadesuit, \mathsf{nil})), \mathsf{nil})$ should **not** be provable from your definition, because the deck in the second position has both a red and a black card.

Similarly, $\mathsf{separate}(\mathsf{cons}(\heartsuit, \mathsf{cons}(\diamondsuit, \mathsf{nil})), \mathsf{cons}(\diamondsuit, \mathsf{cons}(\heartsuit, \mathsf{nil})), \mathsf{nil})$ should not be provable from your definitions, because ordering is not preserved.

Task 2.5 (5 pts). Hopefully, your definition of separate will have a similar property to unshuffle. That is, for any s1 there exists s_2 and s_3 so that $\mathsf{separate}(s_1, s_2, s_3)$ holds. However, it should satisfy a stronger property: for any s_1 the corresponding s_2 and s_3 should be unique. Argue why this is the case. Why does unshuffle *not* have this property?

3 Cutting cards

For this part of the assignment we will define, using simultaneous inductive definition, decks of cards with even or odd numbers of cards in them.

$$\frac{}{\mathsf{nil}\ \mathsf{even}}\ (10) \qquad \frac{c\ \mathsf{card} \quad s\ \mathsf{odd}}{\mathsf{cons}(c, s)\ \mathsf{even}}\ (11) \qquad \frac{c\ \mathsf{card} \quad s\ \mathsf{even}}{\mathsf{cons}(c, s)\ \mathsf{odd}}\ (12)$$

This inductive definition is *simultaneous* (because it simultaneously defines even and odd) as well as *iterated* (because it relies on the previously-defined definition of card).

Task 3.1 (6 pts). What is the induction principle for these judgments? You may want to examine the induction principle for even and odd natural numbers from PFPL.

Task 3.2 (15 pts). Prove well-formedness for the even judgment. That is, prove "For all s, if s even then s deck."

You should use the induction principle from the previous task. Again, be sure to identify what property or properties you are proving with that induction principle.

Task 3.3 (10 pts). Prove the following theorem:

For all S, if

1. $S(\mathsf{nil})$.
2. For all c_1, c_2, and s, if c_1 card, c_2 card, and $S(s)$, then $S(\mathsf{cons}(c_1, \mathsf{cons}(c_2, s)))$.

then for all s, if s even then $S(s)$.

You will want to use the induction principle mentioned above in order to prove this; as always, remember to carefully consider and state the induction hypothesis you are using.

Note: this is a difficult proof, because the induction hypothesis is not immediately obvious. Here's a hint: because you are dealing with a simultaneous inductive definition, the induction hypothesis will have two parts. In our solution, the induction hypothesis pertaining to even-sized decks is "$S(s)$," and the one pertaining to odd-size decks is "For all c', if c' card then $S(\mathsf{cons}(c', s))$."

Proving this statement justifies a new induction principle, a *derived induction principle*:

To show that $\mathcal{S}(s)$ whenever s even, it is enough to show

- $\mathcal{S}(\mathsf{nil})$
- $\mathcal{S}(\mathsf{cons}(c_1, \mathsf{cons}(c_2, s)))$, assuming c_1 card, c_2 card, and $\mathcal{S}(s)$

Task 3.4 (15 pts). Another "operation" on cards is *cutting*, where a player separates a single deck of cards into two decks of cards by removing some number of cards from the top of the deck. We can define cutting cards using an inductive definition.

$$\frac{s\ \mathsf{deck}}{\mathsf{cut}(s, s, \mathsf{nil})}\ (13) \qquad \frac{c\ \mathsf{card} \quad \mathsf{cut}(s_1, s_2, s_3)}{\mathsf{cut}(\mathsf{cons}(c, s_1), s_2, \mathsf{cons}(c, s_3))}\ (14)$$

Using the derived induction principle from the previous task (you can use the induction principle from the previous task even if you do not do the previous task!), prove the following:

For all s_1, s_2, s_3, if s_2 even, s_3 even, and $\mathsf{cut}(s_1, s_2, s_3)$, then s_1 even.

You are allowed to assume the following lemmas:

- **Inversion for nil:** For all s_1 and s_2, if $\mathsf{cut}(s_1, s_2, \mathsf{nil})$, then $s_1 = s_2$ and s_1 deck.
- **Inversion for cons:** For all s_1, s_2, and s_3, if $\mathsf{cut}(s_1, s_2, \mathsf{cons}(c, s_3))$, then there exists a s_1' such that $s_1 = \mathsf{cons}(c, s_1')$, c card, and $\mathsf{cut}(s_1', s_2, s_3)$.

4 Extra Credit: Missing Cards

For the final part of the assignment we will define a way of modelling a deck that is *missing* several cards using generic and hypothetical judgments. Consider the following operator $\mathsf{old}(c_1.c_2.\_)$. It takes a single argument which binds two terms. We have a new judgment od old deck which will be used to define what it means to be an old deck.

$$\frac{c_1, c_2 \mid c_1\ \mathsf{card},\ c_2\ \mathsf{card} \vdash s\ \mathsf{deck}}{\mathsf{old}(c_1.c_2.s)\ \ \mathsf{old\ deck}}$$

With this rule, we stipulate that something is an old deck if for whatever pair of cards we choose to insert into s the result is a valid deck.

Task 4.1 (5 pts). Define (only!) one inference rule for the judgment $od\ c_1\ c_2\ s\ \mathsf{fix}$ which takes an old deck and two cards and "fixes" the old deck by inserting the two new cards into the slots left by the missing cards producing a normal deck s.

Task 4.2 (5 pts). Define one inference rule for the judgment $s\ c_1\ c_2\ od\ \mathsf{remove}$ so that if s is a deck with two cards then od is the old deck version of s with c_1 and c_2 removed. For simplicity (and the next task) ensure that c_1 and c_2 are the top two items on s.

Task 4.3 (5 pts). Assuming that you have completed the previous two tasks, you may now justify that you have done so correctly by proving that they are inverses of sorts. That is prove

> For all s, c_1, c_2, and od so c_1 card, c_2 card, s deck and od old deck holds, then if
>
> $$\mathsf{cons}(c_1, \mathsf{cons}(c_2, s))\ c_1\ c_2\ od\ \mathsf{remove}$$
>
> holds, so does
>
> $$od\ c_1\ c_2\ \mathsf{cons}(c_1, \mathsf{cons}(c_2, s))\ \mathsf{fix}$$

Hint: be sure to use induction on the derivation of $\mathsf{cons}(c_1, \mathsf{cons}(c_2, s))\ c_1\ c_2\ od\ \mathsf{remove}$! This will tell you enough about the structure of all the different arguments to the judgment to prove the claim.