$\boxed{n \text{ nat}}$

natural numbers

$\underbrace{n ::= z \mid s(n)}_{\text{BNF}}$

$$\underbrace{\frac{}{z \text{ nat}}\,z \qquad \frac{n \text{ nat}}{s(n) \text{ nat}}\,s}_{\text{JUDGEMENT}}$$

(inductive relation, here, a predicate.)

$\boxed{+(n_1, n_2) \Downarrow n_3}$

$$\frac{}{+(z, n_2) \Downarrow n_2}\,+z$$

$$\frac{+(n_1', n_2) \Downarrow n_3'}{+(s(n_1'), n_2) \Downarrow s(n_3')}\,+s$$

Function
Total Function
Partial

Lemma: $\forall n_1, n_2$
$(n_1 \text{ nat}) \wedge (n_2 \text{ nat}) \Rightarrow \exists n_3.$ $(n_3 \text{ nat})$
$\wedge\ \underline{+(n_1, n_2) \Downarrow n_3}$

"and" ($\wedge$)

$$\cancel{\frac{}{+(n_1, n_2) \Downarrow n_3}\,+?}$$

pf by ind on $\mathcal{E}$ the der. tree for n_1 nat

case $\dfrac{}{z \text{ nat}}\, z$

- $\exists n_3 = n_2$.
- n_2 nat $\Rightarrow$ n_3 nat
- $+(z, n_2) \Downarrow n_3$ by +Z

---

case $\dfrac{n_1' \text{ nat}}{\underbrace{S(n_1')}_{n_1} \text{ nat}}\, S$

- $n_1 = S(n_1')$
- $(n_1', n_2) \Downarrow n_3'$ by IH
- $+(n_1, n_2) \Downarrow n_3$ by +S with $n_3 = S(n_3')$

$$\boxed{\begin{array}{l} 1.\ +(n_1, n_2) \Downarrow n_3 \\ 2.\ +(n_1, n_2) \Downarrow n_3' \\ \Longrightarrow\ n_3 = n_3' \end{array}}$$

$+(\cdot, \cdot)$ is a function

pf: by ind over (der of) $+(n_1, n_2) \Downarrow n_3$:

Case Z rule:

$$\frac{}{+(Z, n_2) \Downarrow n_2}\,Z$$

$n_3' = \underbrace{n_3 = n_2}_{\text{given}}$

by inversion of

and $+(n_1, n_2) \Downarrow n_3'$

Case S:

$$\frac{+(n_1', n_2) \Downarrow n_3''}{+(\underbrace{S(n_1')}_{n_1}, n_2) \Downarrow \underbrace{S(n_3'')}_{n_3}}\ S$$

want
$n_3 = n_3'$

- $+(n_1', n_2) \Downarrow n_3'''$ by inv of der # 2
- $n_3'' = n_3'''$ by IH
- $S(n_3'') = S(n_3''')$ (b/c S is a function)

$$x = y \implies f(x) = f(y)$$

by IH

$$=$$

$$\dfrac{\dfrac{\vdots}{+(n_1', n_2) \Downarrow n_3''}}{+(S(n_1'), n_2) \Downarrow \underbrace{S(n_3'')}_{n_3}} \qquad\qquad \dfrac{\dfrac{\vdots}{+(n_1', n_2) \Downarrow n_3'''}}{+(S(n_1'), n_2) \Downarrow \underbrace{S(n_3''')}_{n_3'}}$$

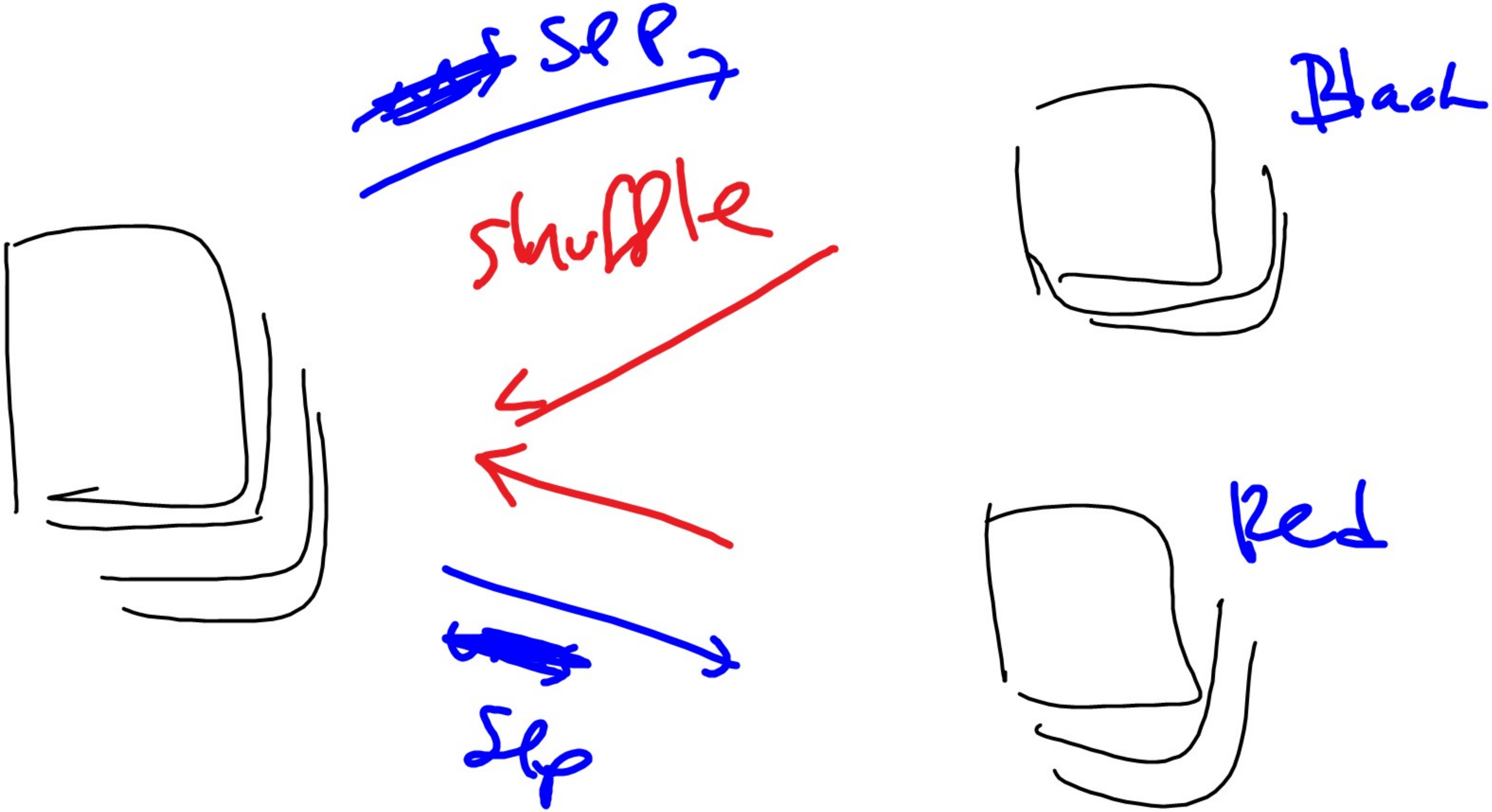
$n ::= z \mid s(n)$ S

$s ::= nil \mid cons(c, s)$ Cons

functions f: $x = y \Rightarrow f(x) = f(y)$

Injective: functions $f(x) = f(y) \Rightarrow x = y$

$s(n_1) = s(n_2) \Rightarrow n_1 = n_2$


shuffle
Black
Red

$[0, 2, 8]$

$[1, 3]$

shuf $\rightarrow$ $[0, 1, 3, \ldots]$

$[0, 2, 8, 1, 3)$

$\vdots$

$+(\cdot , \cdot) \Downarrow \cdot$

$0, 1 \Downarrow 1$

$1, 0 \Downarrow 1$

$\forall s.$

Thm.
$$\begin{cases} 1.\ s \text{ even} \Rightarrow S(s) \\ 2.\ s \text{ odd} \Rightarrow T(s) \end{cases}$$

$T(s) =$
$\forall c'\ c' \text{ card}$
$\Rightarrow S(c' :: s)$

pf by induction:

Case 10:

Case 11: $\left(\dfrac{c \text{ card} \quad s' \text{ odd}}{\underbrace{c :: s'}_{s} \text{ even}}\ 11 \right)$

Case 12:

11

s' odd .. (given)

c card .. (given)

$T(s')$ by IH #2

$S(c :: s')$ by T def.

$S(s)$ $s = c :: s'$

$$\underbrace{1 :: 2 :: 3 :: nil}_{s_1}$$

suffix prefix

$$\underbrace{2, 3}_{s_2} \qquad \underbrace{1}_{s_3}$$